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FEKETE-SZEGÖ INEQUALITY FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS

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ABSTRACT: We introduce some classes of analytic functions, its subclasses and obtain sharp upper bounds of the functional $|a_3 - \mu a_2^2|$ for the analytic function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n, |z| < 1$ belonging to these classes and subclasses.

KEYWORDS: Univalent functions, Starlike functions, Close to convex functions and bounded functions.

MATHEMATICS SUBJECT CLASSIFICATION: 30C50

1. Introduction : Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1.1)$$

which are analytic in the unit disc $\mathbb{E} = \{z: |z| < 1|\}$. Let $\mathcal{S}$ be the class of functions of the form (1.1), which are analytic univalent in $\mathbb{E}$.

In 1916, Bieber Bach ([1], [2]) proved that $|a_2| \le 2$ for the functions $f(z) \in \mathcal{S}$. In 1923, Löwner [10] proved that $|a_3| \le 3$ for the functions $f(z) \in \mathcal{S}$..

With the known estimates $|a_2| \le 2$ and $|a_3| \le 3$, it was natural to seek some relation between a_3 and ${a_2}^2$ for the class $\mathcal{S}$, Fekete and Szegö[4] used Löwner's method to prove the following
well known result for the class $\mathcal{S}$.

Let $f(z) \in \mathcal{S}$, then

$$|a_3 - \mu a_2^2| \le \begin{cases} 3 - 4\mu, if\ \mu \le 0; \\ 1 + 2\exp\left(\frac{-2\mu}{1-\mu}\right), if\ 0 \le \mu \le 1; \\ 4\mu - 3, if \mu \ge 1. \end{cases} \qquad (1.2)$$

The inequality (1.2) plays a very important role in determining estimates of higher coefficients for some sub classes $\mathcal{S}$ ([3], [9]).

Let us define some subclasses of $\mathcal{S}$.

We denote by S*, the class of univalent starlike functions

$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$ and satisfying the condition

$$Re\left(\frac{zg(z)}{g(z)}\right) > 0, z \in \mathbb{E}. \qquad (1.3)$$

We denote by $\mathcal{K}$, the class of univalent convex functions

$h(z) = z + \sum_{n=2}^{\infty} c_n z^n, z \in \mathcal{A}$ and satisfying the condition

$$Re\frac{((zh'(z))}{h'(z)} > 0, z \in \mathbb{E}. \qquad (1.4)$$

A function $f(z) \in \mathcal{A}$ is said to be close to convex if there exists $g(z) \in S^*$ such that

$$Re\left(\frac{zf'(z)}{g(z)}\right) > 0, z \in \mathbb{E}. \qquad (1.5)$$

The class of close to convex functions is denoted by C and was introduced by Kaplan [7] and it was shown by him that all close to convex functions are univalent.

$$S^*(A,B) = \left\{f(z) \in \mathcal{A}; \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}, -1 \le B < A \le 1, z \in \mathbb{E}\right\} \qquad (1.6)$$

$$\mathcal{K}(A,B)=\left\{f(z)\in\mathcal{A};\frac{\left(zf'(z)\right)'}{f'(z)}\prec\frac{1+Az}{1+Bz},-1\le B<A\le 1,z\in\mathbb{E}\right\} \quad (1.7)$$

It is obvious that $S^*(A,B)$ is a subclass of S^* and $\mathcal{K}\ (A,B)$ is a subclass of $\mathcal{K}$.

We introduce a new subclass as $\left\{f(z)\in\mathcal{A};(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right)^{\beta}+\alpha\left(\frac{\left(zf'(z)\right)'}{f'(z)}\right)^{1-\beta}\prec\frac{1+z}{1-z};z\in\mathbb{E}\right\}$ and we will denote this class as $S^*(f,f',\alpha,\beta)$.

We will deal with two subclasses of $S^*(f,f',\alpha,\beta)$ defined as follows in our next paper:

$$S^*(f,f',\alpha,\beta,A,B)=\left\{f(z)\in\mathcal{A};(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right)^{\beta}+\alpha\left(\frac{\left(zf'(z)\right)'}{f'(z)}\right)^{1-\beta}\prec\frac{1+Az}{1+Bz};z\in\mathbb{E}\right\}$$

(1.8)

$$S^*(f,f',\alpha,\beta,\delta)=\left\{f(z)\in\mathcal{A};(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right)^{\beta}+\alpha\left(\frac{\left(zf'(z)\right)'}{f'(z)}\right)^{1-\beta}\prec\left(\frac{1+z}{1-z}\right)^{\delta};z\in\mathbb{E}\right\} \quad (1.9)$$

Symbol $\prec$ stands for subordination, which we define as follows:

Principle of Subordination: Let $f(z)$ and $F(z)$ be two functions analytic in $\mathbb{E}$. Then $f(z)$ is called subordinate to F(z) in $\mathbb{E}$ if there exists a function $w(z)$ analytic in $\mathbb{E}$ satisfying the conditions $w(0)=0$ and $|w(z)|<1$ such that $f(z)=F(w(z))$; $z\in\mathbb{E}$ and we write $f(z)\prec F(z)$.

By $\mathcal{U}$, we denote the class of analytic bounded functions of the form $w(z)=\sum_{n=1}^{\infty}d_n z^n, w(0)=0, |w(z)|<1$.

(1.10)

It is known that $|d_1|\le 1, |d_2|\le 1-|d_1|^2$. (1.11)

2. **PRELIMINARY LEMMAS:** For $0<c<1$, we write $w(z)=\left(\frac{c+z}{1+cz}\right)$ so that

$$\frac{1+w(z)}{1-w(z)}=1+2cz+2z^2+\cdots. \quad (2.1)$$

3. MAIN RESULTS

THEOREM 3.1: Let $f(z)\in S^*(f,f',\alpha,\beta)$., then

$$|a_3-\mu a_2^2|\le\begin{cases}\frac{1}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}\left[\frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{(3\alpha+\beta-4\alpha\beta)}-4\mu\right], & if\mu\le A; \quad (3.1)\\ \frac{1}{3\alpha+\beta-4\alpha\beta} & if A\le\mu\le\boldsymbol{B}; \quad (3.2)\\ \frac{1}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}\left[4\mu-\frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{(3\alpha+\beta-4\alpha\beta)}\right], & if\ \mu\ge B (3.3)\end{cases}$$

Where $A=\frac{8\alpha+3\beta+4\alpha^2-\beta^2-3\alpha\beta^2-7\alpha\beta}{4(3\alpha+\beta-4\alpha\beta)}$ and

$$B=\frac{8\alpha+3\beta+8\alpha^2+\beta^2-24\alpha^2\beta-6\alpha\beta^2-7\alpha\beta}{4(3\alpha+\beta-4\alpha\beta)}$$

The results are sharp.

Proof: By definition of $S^*(f,f',\alpha,\beta)$, we have

$$(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right)^{\beta}+\alpha\left(\frac{\left(zf'(z)\right)'}{f'(z)}\right)^{1-\beta}=\frac{1+w(z)}{1-w(z)};w(z)\in\mathcal{U}. \quad (3.4)$$

Expanding the series (3.4), we get

$$(1-\alpha)\left\{1+\beta a_2 z+(2\beta a_3+\frac{\beta(\beta-3)}{2}a_2^2)z^2+---\right\}+\alpha\{1+2(1-\beta)a_2 z+2(1-\beta)(3a_3-(\beta+2)a_2^2)z^2+---\}=(1+2c_1 z+2(c_2+c_1{}^2)z^2+---). \quad (3.5)$$

Identifying terms in (3.5), we get

$$a_2=\frac{2}{(1-\alpha)\beta+2\alpha(1-\beta)}\,c_1 \quad (3.6)$$

$$a_3=\frac{1}{3\alpha+\beta-4\alpha\beta}\,c_2+\frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{(3\alpha+\beta-4\alpha\beta)\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}\,c_1{}^2. \quad (3.7)$$

From (3.6) and (3.7), we obtain

$$a_3-\mu a_2^2=\frac{1}{3\alpha+\beta-4\alpha\beta}c_2+\left[\frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{(3\alpha+\beta-4\alpha\beta)\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}-\frac{4}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}\mu\right]c_1^2. \quad (3.8)$$

Taking absolute value, (3.8) can be rewritten as

$$|a_3-\mu a_2^2|\le\frac{1}{3\alpha+\beta-4\alpha\beta}|c_2|+\frac{1}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}\left|\frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{(3\alpha+\beta-4\alpha\beta)}-4\mu\right||c_1^2|. \quad (3.9)$$

Using (1.9) in (3.9), we get

$$|a_3-\mu a_2^2|\le\frac{1}{3\alpha+\beta-4\alpha\beta}(1-|c_1|^2)+\frac{1}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}\left|\frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{(3\alpha+\beta-4\alpha\beta)}-4\mu\right||c_1^2|=$$

$$\frac{1}{3\alpha+\beta-4\alpha\beta}+\frac{1}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}\left[\left|\frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{(3\alpha+\beta-4\alpha\beta)}-4\mu\right|-\frac{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}{3\alpha+\beta-4\alpha\beta}\right]|c_1|^2. \quad (3.10)$$

Case I: $\mu\le\frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{4(3\alpha+\beta-4\alpha\beta)}$. (3.10) can be rewritten as

$$|a_3-\mu a_2^2\le\frac{1}{3\alpha+\beta-4\alpha\beta}+\frac{1}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}\left[\frac{8\alpha+3\beta+4\alpha^2-\beta^2-3\alpha\beta^2-7\alpha\beta}{(3\alpha+\beta-4\alpha\beta)}-4\mu\right]|c_1|^2. \quad (3.11)$$

Subcase I (a): $\mu\le\frac{8\alpha+3\beta+4\alpha^2-\beta^2-3\alpha\beta^2-7\alpha\beta}{4(3\alpha+\beta-4\alpha\beta)}$. Using (1.9), (3.11) becomes

$$|a_3-\mu a_2^2\le\frac{1}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}\left[\frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{(3\alpha+\beta-4\alpha\beta)}-4\mu\right]. \quad (3.12)$$

Subcase I (b): $\mu\ge\frac{8\alpha+3\beta+4\alpha^2-\beta^2-3\alpha\beta^2-7\alpha\beta}{4(3\alpha+\beta-4\alpha\beta)}$. We obtain from (3.11)

$$|a_3-\mu a_2^2\le\frac{1}{3\alpha+\beta-4\alpha\beta}. \quad (3.13)$$

Case II: $\mu\ge\frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{4(3\alpha+\beta-4\alpha\beta)}$

Preceding as in case I, we get

$$|a_3-\mu a_2^2\le\frac{1}{3\alpha+\beta-4\alpha\beta}+\frac{1}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}\left[4\mu-\frac{8\alpha+3\beta+8\alpha^2+\beta^2-24\alpha^2\beta-6\alpha\beta^2-7\alpha\beta}{(3\alpha+\beta-4\alpha\beta)}\right]|c_1|^2. \quad (3.14)$$

Subcase II (a): $\mu\le\frac{8\alpha+3\beta+8\alpha^2+\beta^2-24\alpha^2\beta-6\alpha\beta^2-7\alpha\beta}{4(3\alpha+\beta-4\alpha\beta)}$

(3.14) takes the form $|a_3 - \mu a_2^2 \leq \frac{1}{3\alpha+\beta-4\alpha\beta}$ (3.15)

Combining subcase I (b) and subcase II (a), we obtain

$$|a_3 - \mu a_2^2| \leq \frac{1}{3\alpha+\beta-4\alpha\beta} \, if \, \frac{8\alpha+3\beta+4\alpha^2-\beta^2-3\alpha\beta^2-7\alpha\beta}{4(3\alpha+\beta-4\alpha\beta)} \leq \mu \leq \frac{8\alpha+3\beta+8\alpha^2+\beta^2-24\alpha^2\beta-6\alpha\beta^2-7\alpha\beta}{4(3\alpha+\beta-4\alpha\beta)} \quad (3.16)$$

Subcase II (b): $\mu \geq \frac{8\alpha+3\beta+8\alpha^2+\beta^2-24\alpha^2\beta-6\alpha\beta^2-7\alpha\beta}{4(3\alpha+\beta-4\alpha\beta)}$

Preceding as in subcase I (a), we get

$$|a_3 - \mu a_2^2| \leq \frac{1}{\{(1-\alpha)\beta+2\alpha(1-\beta)\}^2}\left[4\mu - \frac{8\alpha+3\beta+4\alpha^2-12\alpha^2\beta-9\alpha\beta^2-7\alpha\beta}{(3\alpha+\beta-4\alpha\beta)}\right]. \quad (3.17)$$

Combining (3.12), (3.16) and (3.17), the theorem is proved.

Extremal function for (3.1) and (3.3) is defined by

$f_1(z) = (1 + az)^b$

Where

$$a = \frac{\{(2\alpha+\beta-3\alpha\beta)^2-(1-\alpha)\beta(\beta-3)+4\alpha(1-\beta)(\beta+2)\}a_2^2-4(3\alpha+\beta-4\alpha\beta)a_3}{(2\alpha+\beta-3\alpha\beta)a_2}$$

And

$$b = \frac{(2\alpha+\beta-3\alpha\beta)^2 a_2^2}{\{(2\alpha+\beta-3\alpha\beta)^2-(1-\alpha)\beta(\beta-3)+4\alpha(1-\beta)(\beta+2)\}a_2^2-4(3\alpha+\beta-4\alpha\beta)a_3}$$

Extremal function for (3.2) is defined by $f_2(z) = z(1 + Bz^2)^{\frac{A-B}{2B}}$.

Corollary 3.2: Putting $\alpha = 1, \beta = 0$ in the theorem, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} 1-\mu, if \mu \leq 1; \\ \frac{1}{3} if 1 \leq \mu \leq \frac{4}{3}; \\ \mu - 1, if\ \mu \geq \frac{4}{3} \end{cases}$$

These estimates were derived by Keogh and Merkes [8] and are results for the class of univalent convex functions.

Corollary 3.3: Putting $\alpha = 0, \beta = 1$ in the theorem, we get

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3-4\mu, if \mu \leq \frac{1}{2}; \\ 1 if \frac{1}{2} \leq \mu \leq 1; \\ 4\mu - 3, if\ \mu \geq 1 \end{cases}$$

These estimates were derived by Keogh and Merkes [8] and are results for the class of univalent starlike functions.

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